

## Situations Project

Situation: Using curriculum analysis as a vehicle for exploring and learning mathematics

Ira J Papick

University of Nebraska-Lincoln

### **Prompt**

The following investigation concerning pattern type problems occurred in some professional development experiences with middle and secondary teachers (as well as with some doctoral students in mathematics education). The participants involved in this investigation worked in small groups and were asked to mathematically critique (with the aid of a basic qualitative framework) the following sampling of pattern type problems that appear in middle grade curricula and on state and national assessments. (For the purposes of this Situation, we will not discuss the analysis of all sample problems listed below. For a complete analysis, see (Papick, I, Regis, T, and Olson, T, 2008).

Sample Problems:

1. For each sequence of integers given below, determine the next three terms in the sequence and record your results in a table (see the start of such a table for problem (a) below). Write a rule for finding the  $n^{\text{th}}$  term of the sequence, and justify why your rule is correct. Use your rule to determine if the number 79 appears in the sequence, and if it does, what term would 79 be in the list?

a) 3, 5, 7,...

Position in the list	Term in the list
1	3
2	5
3	7
4	
5	
6	

b) 4, 7, 12, 19, 28,...

c) 2, 4, 8, 16,...

d) 1,1, 2, 3, 5, 8, 13,...

2. If a stock is worth \$2.25 on Monday, and it's value increases to \$2.50 on Tuesday, and \$2.75 on Wednesday, then what will its value be on Thursday and Friday?
3. What positive integers can arise as the sum of four consecutive positive integers? List the five smallest sums that are possible in increasing magnitude, and find a rule for the  $n^{\text{th}}$  term of your list.

4. Consider the sequence of dot diagrams:



Assume that the number of dots added at each step is more than the number added in the previous step. How many dots in the 20<sup>th</sup> term?

5. Determine the number of distinct non-overlapping regions that can be created by connecting  $n$  positive integer points on a circle with chords, such that not more than two chords pass through any interior point of the circle (this last condition is included so that the maximum number of regions will occur for each  $n$ ).

### **Commentary**

The critical analysis was to focus on two main questions:

- Are the sample problems mathematically sound?
- Do the sample problems meet the desired learning objectives?

*Mathematical soundness:* For the purposes of this investigation, we adopted (with slight modifications) the following defining characteristics, utilized by Hyman Bass (2005) in his Brookings Institution assessment analysis of selected Algebra items from the NAEP instrument, to evaluate the mathematical soundness of the given pattern-type problems:

- (a). A problem should be clearly and precisely formulated at a level accessible to the intended grade, and free of non-purposeful ambiguity.
- (b). A problem should involve substantial mathematical knowledge and/or skills.
- (c). If a problem is contextualized, it should have a sensible context that is taken seriously and one that does not obscure the intended mathematical focus of the item.

*Learning objectives:* The stated or intended learning objectives usually associated with these kinds of problems (determined either by teacher materials or the explicit problem instructions) are for students to recognize patterns and represent them with tables, graphs, and symbolic notation.

### **Mathematical Foci**

#### *Mathematical Focus 1*

#### *A mathematical critique of Sample problem 1.*

Although the given lists only display a finite set of numbers, the notation used and the questions posed indicate that each list is assumed to continue on indefinitely governed by an inherent structure, and that the remaining terms can be determined by some specific rule related to that structure. In particular, the nature of these problems implies that each

given list is understood to be a *sequence*, i.e., a function defined from the positive integers into any nonempty set. Since, by definition, the domain of a sequence  $f$  is the positive integers, it is standard practice to represent the function  $f$  in the form  $a_1, a_2, a_3, \dots, a_n, \dots$ , where  $f(n) = a_n$  for each positive integer  $n$ .

When a student first encounters a problem of the type just described, their focus (which is influenced by teacher and curriculum) is on finding the *expected* solution, rather than thinking about if the given problem makes mathematical sense, or if the provided information actually leads to the anticipated result. A usual opening strategy is for them to calculate changes between the successive terms of the given sequence and then attempt to use those changes to describe subsequent terms. For example, in the list 3, 5, 7, ..., a student might proceed by observing that each succeeding term is two more than the preceding term, and thus the fourth term should be 9, the fifth term 11, and the sequence can be represented in general by the rules,  $a_1 = 3$  and  $a_n = a_{n-1} + 2$ . This kind of recursive reasoning and representation approach is a frequent strategy employed by children when initially working with elementary pattern problems, rather than finding a closed form formula to represent the sequence, i.e., finding an explicit rule  $f$  expressing the  $n^{\text{th}}$  member of the sequence in terms of the positive integer  $n$  (in this case,  $f(n) = 2n + 1$ ).

Although it is common for curricular materials, teachers, and students to declare that 9 is the unique answer to the question: “What is the fourth term in the sequence 3, 5, 7, ...?”, it is in fact an ill-posed question that does not have a unique answer. For example, one student might say the answer is 11, since they thought the list was of all odd prime numbers, whereas another student might say the next term is 3, since they thought the sequence was periodic and just repeated the first three terms forever. These are just two conceivable answers derived from the infinitely many possible sequences  $f$  (functions  $f$  defined on the positive integers), where  $f(1) = 3, f(2) = 5$ , and  $f(3) = 7$ . Without knowing more about the defining properties of the sequence  $f$  or some underlying structure that the sequence emerged from, it is simply speculation (or in the vernacular of children, “a guessing game”) to determine  $f(n)$  for any  $n \geq 3$ , rather than a precise mathematical deduction. In summary, we would classify this problem as one that is not mathematically sound, since it fails to meet criterion (a), and to some extent criterion (b).

Since a principle purpose of these sequence problems (as stated in state-wide and national standards documents and curriculum learning expectation guides) is to introduce students to fundamental algebraic concepts (such as variables and functions), and algebraic procedures (such as solving equations), it is crucial to present the problems as coherent well-defined mathematical statements that support the desired learning goals. For example, one uncomplicated reformulation of the previous problem into more precise mathematical language, and one that meets the intended algebraic learning objectives, is as follows:

*Restated problem.* Find a rule for a sequence whose first three terms are 3, 5, 7, and then use that rule to determine the fourth, fifth and sixth terms of the sequence. Another rephrasing of the problem is to find a rule that produces the below table, and then use that rule to expand the table to one that contains positions 4, 5, 6, etc.

Position in the list	Term in the list
1	3
2	5
3	7

### *Mathematical Focus 2*

*There are infinitely many polynomial functions that agree with the previous table (i.e.,  $g(1) = 3$ ,  $g(2) = 5$ ,  $g(3) = 7$ ) (In accordance with the standard convention, we will use the notation  $g(x)$  to represent the polynomial, and  $g$  to represent its induced polynomial function). In particular, a polynomial function  $g$  induced by the polynomial  $g(x)$  (for our purposes having rational number coefficients) satisfies  $g(1) = 3$ ,  $g(2) = 5$ , and  $g(3) = 7$  if and only if the polynomial  $g(x)$  is given by  $g(x) = (2x + 1) + (x - 1)(x - 2)(x - 3)h(x)$ , where  $h(x)$  is some polynomial with rational coefficients.*

*The method of producing such polynomials easily extends to any such finite table of values.*

If the polynomial function  $g$  induced by the polynomial  $g(x)$  satisfies  $g(1) = 3$ ,  $g(2) = 5$ , and  $g(3) = 7$ , then the polynomial function  $k$  induced by the polynomial  $k(x) = g(x) - (2x + 1)$  satisfies  $k(1) = 0$ ,  $k(2) = 0$ , and  $k(3) = 0$ . Thus, by the Factor Theorem,  $k(x) = (x - 1)(x - 2)(x - 3)h(x)$ , where  $h(x)$  is some polynomial with rational coefficients, and so  $g(x) = (2x + 1) + (x - 1)(x - 2)(x - 3)h(x)$ .

A consequence of the above result is that the only polynomials functions  $g$  that have the values  $g(1) = 3$ ,  $g(2) = 5$ , and  $g(3) = 7$  are induced by polynomials of all degrees other than degree two. Thus a middle grade/secondary teacher who is discussing linear and quadratic polynomials with his/her students, and who is aware of the above observation, might raise the following pertinent question: Given three distinct points on a line, does there exist a quadratic polynomial function whose graph passes through those points? Justify your answer. Raising this question certainly does not depend upon knowing the previous observation, but the classroom investigation connected to this question could be so much richer if a teacher were equipped with this kind of knowledge.

### *Mathematical Focus 3:*

*Creating well-defined pattern problems that lead to unique solutions.*

In the analysis of example 1(a), we offered a restatement for that problem that transformed it into a well-defined mathematical problem, however an alternate way to present pattern-type problems, and one that authentically reflects the way mathematicians study patterns, is to create problems where these or similar sequences emerge from a structured mathematical or “real world” context. For example, consider sample problem 3: What positive integers can arise as the sum of four consecutive positive integers? List the five smallest sums that are possible in increasing magnitude, and find a rule for the  $n^{\text{th}}$  term of your list.

This problem has similar learning goals to sample problem 1 (i.e., recognize a pattern in a list of numbers and represent it by a rule), but it significantly differs from them in that there is a unique (infinite) sequence and corresponding rule that emerges from the mathematical structure of the problem. In particular, consider the following solution:  
 $(1 + 2 + 3 + 4) = 10$ ,  $(2 + 3 + 4 + 5) = 14$ , 18, 22, 26, ...,  $n + (n + 1) + (n + 2) + (n + 3) = 4n + 6$ .

*Mathematical Focus 4:*

*Using counting to analyze a pattern problem.*

Although the dot diagram problem (sample problem 4) as stated does not provide the necessary information to uniquely determine the number of dots in the 20<sup>th</sup> term (hence, we would say this problem and others like it are not mathematically sound, since they have the same deficiencies as the number list problems that we previously analyzed) Milgram (2006) does indicate that the conditions of the problem do lead to a lower bound for the number of dots in the 20<sup>th</sup> term, and in general the  $n^{\text{th}}$  term. He notes that the third term contains 6 more dots than the second term, so the fourth term must contain at least 7 more dots than the third term, and so on. He then concludes that the lower bound for the  $n^{\text{th}}$  term is given by the expression:

$$\frac{(n + 3)(n + 4)}{2} - 9, n \geq 3,$$

and so the lower bound for the 20<sup>th</sup> term, i.e.,  $n = 20$ , is 267 dots. Using the formula for the sum of an arithmetic progression, one can arrive at the lower bound expression obtained by Milgram (2006) in the following manner:

Finding the lower bound: Since the third term of the given dot sequence contains 6 more dots than the second term, i.e., 12 dots, the fourth term contains at least  $12 + 7$  dots, the fifth term contains at least  $12 + 7 + 8$  dots, and thus the  $n^{\text{th}}$  term contains at least  $12 + 7 + 8 + \dots + (n + 3)$  dots. Finally, notice that

$$12 + 7 + 8 + \dots + (n + 3) = [1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + \dots + (n + 3)] - 9 = \frac{(n + 4)(n + 3)}{2} - 9.$$

## References

Papick, I, Regis, T, and Olson, T (2008). Analyzing Numerical and Geometrical Pattern Problems in Middle Grade Mathematics Curricula, *Investigations in Mathematics Learning*, accepted for publication.

Milgram, J. (2006). Pattern recognition problems in K-12, preprint.